Chapter Nine

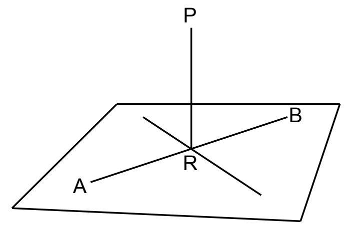
Basic Solid Geometry

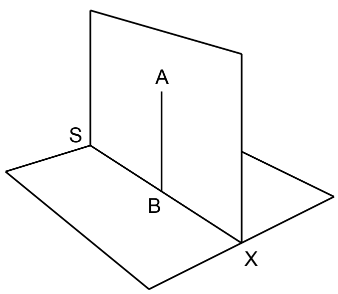
DEFINITIONS

1. A SOLID is whatever has length, width, and depth.

2. You have a STRAIGHT LINE PERPENDICULAR TO A PLANE if it is perpendicular to all the straight lines it stands on in that plane.

For example, PR *stands on* AB, *and is perpendicular to it. And if* PR *is perpendicular to all such lines in the plane passing through* R, *then* PR *is perpendicular to the plane*.

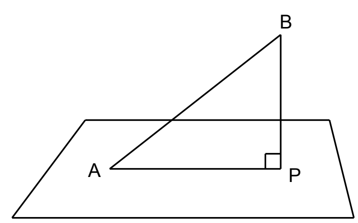


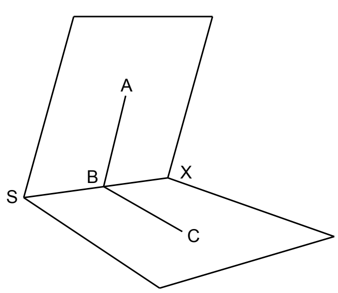


3. You have PERPENDICULAR PLANES if every straight line in one of them that is perpendicular to their intersection is also perpendicular to the other plane.

For example, let two planes intersect along SX. *If every straight line* AB *drawn perpendicular to* SX *in one of the planes is also perpendicular to the other plane, then the two planes are perpendicular to each other.*

4. Consider a straight line AB that passes through some plane at point A, and is not perpendicular to the plane, but leans over somewhat. How much does it lean? If we choose any point B along it, and BP falls perpendicular to the plane, then the INCLINATION OF THE STRAIGHT LINE TO THE PLANE is angle BAP.

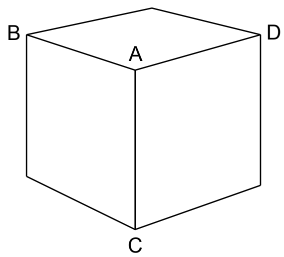




5. The INCLINATION OF A PLANE TO A PLANE is the angle between two straight lines, one drawn in each plane, and both drawn perpendicular to the line of intersection and from the same point on it.

For example, angle ABC *is formed by two perpendiculars to* SX, *each of them drawn in one of the two planes. So angle* ABC *is the inclination of the planes*.

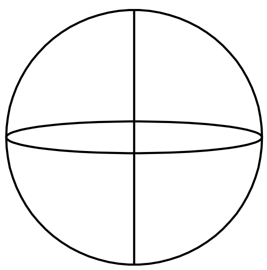
6. PARALLEL PLANES are those which never meet, no matter how far they are extended.



7. A PLANAR SOLID ANGLE is formed by three or more planes meeting at a point.

For example, in a cube, one angle of it is formed by three right-angled faces, namely BAC, BAD, CAD.

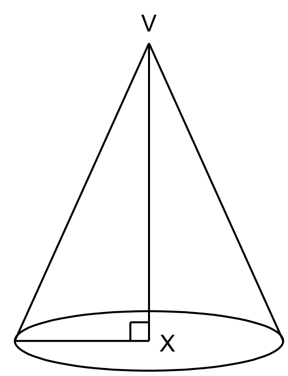
8. A SOLID FIGURE is a figure contained by one or more surfaces.

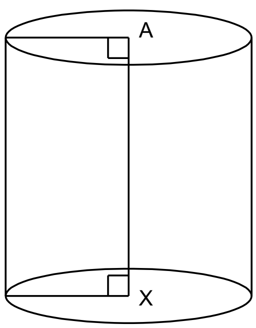


9. A SPHERE is a solid figure contained by one surface which is at all points equidistant from one point within called the CENTER. If a semicircle is rotated all the way around on its diameter once, the solid figure it describes is a sphere. The center and diameter of the semicircle are also the center and diameter of the sphere. Any straight line drawn through the center of a sphere and stopping at the surface of the sphere in each direction is a DIAMETER of the sphere.

10. A RIGHT CONE is a solid figure described by rotating a right triangle all the way around once about one of the sides forming the right angle. The side about which the triangle was rotated is called the AXIS of the cone, and the circle described by the other side of the right angle is the BASE of the cone. The point at which the axis meets the hypotenuse of the original triangle is the VERTEX of the cone.

Note: this kind of cone is called a right *cone because its axis is at right angles to its base. There are other kinds of cones called* oblique *cones, but since these will not come up in this book, the simple term* cone *will always refer to a right cone*.





11. A RIGHT CYLINDER is a solid figure described by rotating a rectangle all the way around once about one of its sides. The side about which the rectangle was rotated is called the AXIS of the cylinder, and the two circles described by the two sides of the rectangle adjacent to the axis are the BASES of the cylinder.

Note: this kind of cylinder is called a right *cylinder because its axis is at right angles to its bases. There are other kinds of cylinders called* oblique *cylinders, but since these will not come up in this book, the simple term* cylinder *will always refer to a right cylinder*.

12. SIMILAR CONES or SIMILAR CYLINDERS are those in which the axes and the diameters of the bases are proportional.

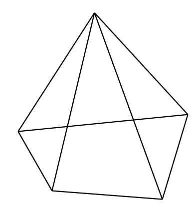
13. A POLYHEDRON is a solid figure contained by four or more rectilineal plane figures. *Note: the plural for* polyhedron *is often written* polyhedra. *I prefer to say* polyhedrons.

14. SIMILAR POLYHEDRONS are those whose faces are similar, each to each, and similarly arranged.

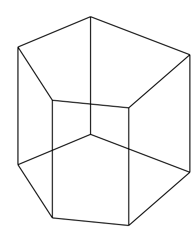
By "similarly arranged" I mean that if any two faces in one solid meet each other, then the two correspondingly similar faces in the other solid also meet each other, forming an edge; also, if a solid angle in one solid is convex, then the corresponding solid angle in the other solid is also convex, but if concave, then concave.

SIMILAR AND EQUAL POLYHEDRONS are similar polyhedrons whose corresponding faces are equal in size. These can also be called *congruent* polyhedrons.

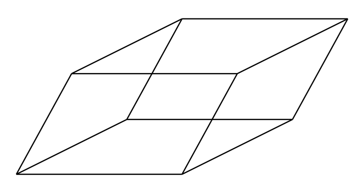
15. A PYRAMID is a polyhedron contained by a plane and three or more triangles drawn down to it from one point. The portion of the plane bounding the pyramid is called its BASE, whereas the point is called its VERTEX.



16. A PRISM is a polyhedron contained by two congruent and parallel polygons similarly oriented, and all the parallelograms joining their corresponding sides. The two identical and parallel polygons are the BASES of the prism.



17. A PARALLELEPIPED is a prism whose bases are parallelograms.



BASIC PRINCIPLES OF SOLID GEOMETRY

1. If any two points of a straight line lie in a plane, the whole straight line lies in that plane.

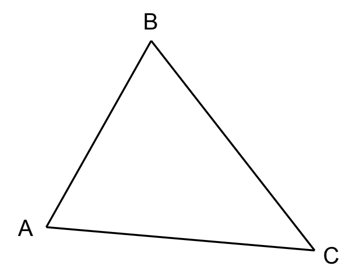
2. If two planes intersect, their intersection is a straight line, and they have no other points in common.

3. Any plane can be extended as far as we please in any of its directions.

4. Any plane can be rotated about any straight line that lies within it.

THEOREMS

THEOREM 1: *Any three points not lying in a straight line lie only in one plane, and every triangle lies only in one plane.*



Consider any three points A, B, C which do not lie in one straight line. It is impossible for all three of these points to lie in more than one plane.

If possible, suppose A, B, C all lie in two distinct planes: plane Q and also plane Z.

Now, since A and B both lie in plane Q, therefore straight line AB lies in plane Q (Princ. 1). And since A and B both lie in plane Z, therefore straight line AB lies in plane Z (Princ. 1). Therefore plane Q and plane Z have line AB in common, i.e. they intersect along that straight line. But then they have no other points in common, beyond those lying in a straight line with AB (Princ. 2). Therefore point C, not lying in line with AB (given), is *not* common to planes Q and Z. And thus it is not possible for A, B, C to lie all in plane Q, and also all in plane Z.

Again, the whole triangle ABC lies only in one plane. For any plane containing all of triangle ABC must also contain its three vertices A, B, and C. But we have just showed that there is only one such plane. Therefore the whole of any triangle lies only in one plane.

Q.E.D.

THEOREM 1 Remarks:

1. One point can have many straight lines passing through it, but any two points lie only in one straight line. Similarly, two points can have many planes passing through them, but any three points (if they are not in a straight line) lie only in one plane.

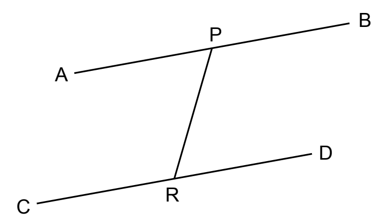
It is obvious that any two points can in fact have a straight line passing through them. Is it also obvious that any three points can have a plane passing through them? Yes. Say the points are A, B, C. Join AB, and pass any plane through AB. Now rotate the plane around AB like a hinge until it hits C, and the result is a plane containing points A, B, and C.

So any three points do lie in one plane. But a fourth point might not lie in the same plane.
2. Obviously, if you have 3 points in a straight line, there is an infinity of planes that contain those 3 points. Pass any plane through the straight line containing the 3 points, and this plane will contain all 3 points.

3. If it were not obvious enough by itself, it is now obvious that *One and only one plane can be drawn through a given straight line and a given point not on that straight line*. For example, only one plane goes through straight line AB and point C – otherwise, more than one plane would contain the three points A, B, C.

4. If it were not obvious enough by itself, it is now obvious that *One and only one plane can be drawn through a given pair of intersecting straight lines*. For example, only one plane goes through the straight lines AB and BC – otherwise, more than one plane would contain the three points A, B, C.

THEOREM 2: *One and only one plane passes through any pair of parallel straight lines, and any straight line joining any two points on the parallels also lies in that plane*.



Given: AB and CD, a pair of parallel straight lines, with P and R being random points on each of them.

Prove: One and only one plane passes through both AB and CD, and PR lies in that plane.

Put a pencil down on the table, and imagine it indicating a straight line going on forever in both directions, say North and South. Now hold a pen over the pencil, but pointing East and West. These two straight lines will never intersect each other, and yet we do not call them "parallel." Why? Because they are not in the same plane. It is especially interesting that even in the same plane two straight lines can be so oriented that they will never meet – there is in fact only *one* orientation you can give a straight line to make it parallel to another. And thus "parallel" means not only "never meeting," but also "in one plane." Therefore the first part of the theorem, namely that any two straight lines that are parallel must lie in the same plane, is really self-evident. It is part of what "parallel" means.

It is also clear that the parallels AB and CD lie *only* in one plane – it is not possible for more than one plane to contain them both. For supposing it were so, then two distinct planes would contain points A, B, C, even though these do not lie in a straight line with each other, which is impossible (Thm.1). Thus it is impossible for more than one plane to contain a given pair of parallel straight lines.

And since points P and R both lie in the plane containing the parallels AB and CD, therefore the line PR lies in that plane, too (Princ. 1).

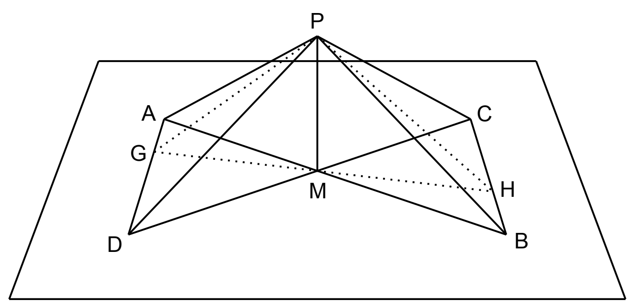
Q.E.D.

THEOREM 2 Remarks:

A pair of lines that never meet, but are *not* in the same plane as each other, are called *skew* lines.

THEOREM 3: *If a straight line is perpendicular to two intersecting straight lines at the point where they meet, then it is perpendicular to the plane in which they lie.*

Suppose AB and CD meet each other at M, and PM is perpendicular to both AB and CD. Then I say PM is perpendicular to the plane passing through AB and CD.



Cut off MA = MD = MB = MC.
Join AD, BC, AP, DP, BP, CP.
In the plane of AB and CD, draw GMH through M at random, cutting AD and BC.
Join GP, HP.

I say that PM is at right angles to the line GMH drawn through M randomly in the plane.

[1] Since ∠AMD and ∠BMC are vertical and are contained by equal lines, hence △MAD ≅ △MBC, so ∠MAD = ∠MBC.

[2]	Now	∠MAG = ∠MBH	(being the same as ∠MAD and ∠MBC)
	but	∠AMG = ∠BMH	(being vertical)
	and	<u>AM = MB</u>	(we made them so)
	so	△MAG ≅ △MBH	(Angle Side Angle)

[3]	Again	PA = PB	(△MAP ≅ △MBP by S-A-S)
	and	PD = PC	(△MDP ≅ △MCP by S-A-S)
	and	<u>AD = BC</u>	(△MAD ≅ △MBC; Step 1)
	so	△PAD ≅ △PBC	(Side Side Side)
	so	∠PAD = ∠PBC	

[4]	Again	∠PAG = ∠PBC	(being the same as ∠PAD and ∠PBC)
	and	PA = PB	(△MAP ≅ △MBP)
	and	<u>AG = BH</u>	(△MAG ≅ △MBH; Step 2)

so $\triangle PAG \cong \triangle PBH$ (Side Angle Side)

[5] Now $PG = PH$ ($\triangle PAG \cong \triangle PBH$; Step 4)
and $MG = MH$ ($\triangle MAG \cong \triangle MBH$; Step 2)
and <u>PM is common</u> (to $\triangle PMG$ and $\triangle PMH$)
so $\triangle PMG \cong \triangle PMH$ (Side Side Side)
so $\angle PMG = \angle PMH$

But these equal angles are adjacent. Hence PM is at right angles to GMH.

[6] Since PM is thus at right angles to any straight line drawn through M in the plane of AB and CD, therefore PM is perpendicular to that plane.

Q.E.D.

THEOREM 3 Remarks:

1. If GH is drawn through M so that it does not cut AD and BC, then it will cut AC and DB, and we use them for the proof instead.

2. A kind of converse to this Theorem is: *All perpendiculars to one point on a straight line lie in one plane*. All the perpendiculars to PM drawn from M lie in the plane of AB and CD.

3. Prove that the line drawn perpendicular to a plane from a point above it is the shortest straight line that can be drawn from that point to the plane.

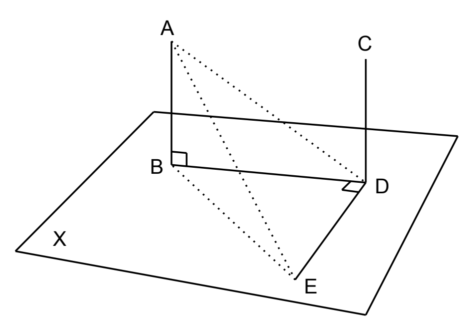
4. Prove that only one straight line can be drawn from a given point perpendicular to a given plane.

THEOREM 4: *If one of two parallels is perpendicular to a plane, so is the other.*

Given: AB is parallel to CD, and AB is perpendicular to plane X.

Prove: CD is also perpendicular to plane X.

Join BD.
Draw DE (in plane X) perpendicular to BD, and cut off DE equal to AB.



Join BE, AE, AD.

[1]	Now,	$\triangle ABD \cong \triangle EDB$	(SAS)
[2]	Thus	AD = BE	(see Step 1)
	but	DE = AB	(we made them equal)
	and	AE is common	(to $\triangle ABE$ and $\triangle EDA$
	so	$\triangle ABE \cong \triangle EDA$	(SSS)
[3]	Now	$\angle ABE = \angle EDA$	(by Step 2)
	but	$\angle ABE$ is right	(since AB is given perpendicular to plane X)
	so	$\angle EDA$ is right	

[4] So ED is perpendicular to DA
but ED is perpendicular to DB (by construction)
so ED is perpendicular to the plane through DA and DB (Thm.3), i.e. the plane containing points A, B, D.

[5] Now, there is only one plane containing points A, B, D (Thm.1), but the plane containing parallels AB and CD (Thm.2) contains points A, B, D, and therefore the plane containing points A, B, D is the same as the plane containing parallels AB and CD. Thus ED is perpendicular to the plane of the parallels, i.e. to the plane containing triangle BDC. Therefore $\angle EDC$ is a right angle (see Def.2).

[6] So CD is perpendicular to DE (Step 5)
and CD is perpendicular to BD ($\angle ABD$ is right, and CD is parallel to AB)
so CD is perpendicular to two straight lines intersecting in plane X, and thus CD is perpendicular to plane X (Thm.3).

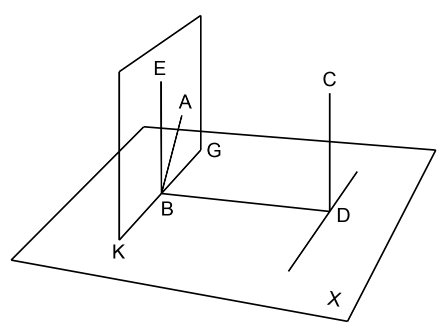
Q.E.D.

THEOREM 5: *If two straight lines are perpendicular to the same plane, they are parallel.*

Given: AB is perpendicular to plane X.
CD is perpendicular to plane X.

Prove: AB is parallel to CD.

Suppose, if possible, that AB is *not* parallel to CD. Then since B, D, C are all in one plane, draw BE in this plane parallel to CD. Therefore BE is perpendicular to plane X (Thm.4).



Now, A, B, E are all in one plane. Let the intersection of their plane with plane X be called GK.

Since BE is perpendicular to plane X, therefore ∠GBE is right.

Since AB is perpendicular to plane X, therefore ∠GBA is right.

Thus ∠GBE = ∠GBA, i.e the whole is equal to the part, which is impossible. Thus our initial assumption was impossible – AB in fact *is* parallel to CD.

Q.E.D.

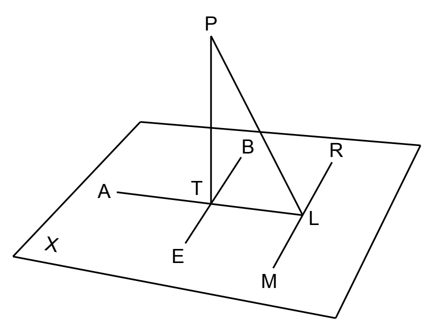
THEOREM 5 Remarks:

From this it is clear that *You can't have two straight lines perpendicular to the same point on a plane* (except, of course, on opposite sides of the plane, i.e. one above it and one below it).

THEOREM 6: *How to drop a straight line perpendicular to a plane from a given point above it.*

Suppose P is the point above our plane X. Choose any straight line RM in plane X. Thus P, R, M are in one and only one plane – drop PL perpendicular to RM in that new plane (as we learned to do in Ch. 1).

Now draw LA perpendicular to RLM in plane X. Thus P, L, A are in one and only one plane. Drop PT perpendicular to LA in that new plane.



I say that PT is perpendicular to plane X.

In plane X, draw BTE parallel to RLM.

Now RLM is perpendicular to plane PLT, since it is perpendicular to both PL and LT by construction (Thm.3). Thus BTE, parallel to RLM, is also perpendicular to plane PLT (Thm.4).

Thus BT is perpendicular to all lines through T in plane PLT (Def.2).

So BT is perpendicular to PT.

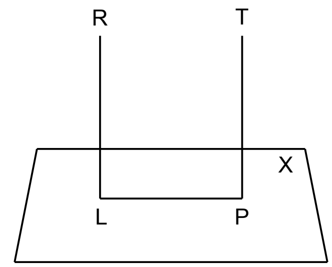
But LT is perpendicular to PT, by construction.

So PT is perpendicular to BT *and* LT, which both lie in plane X.

Therefore PT is perpendicular to plane X (Thm.3).

Q.E.F.

THEOREM 7: *How to set up a straight line perpendicular to a plane from a given point on it.*



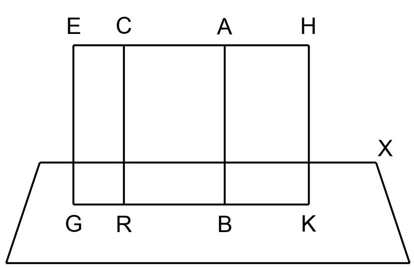
Given: Point P in plane X.
Make: A straight line perpendicular to plane X at P.

Choose any point R at random above plane X, and drop RL perpendicular to plane X (Thm.6).
In the plane of R, L, P draw PT parallel to RL.

Now	RL is perpendicular to plane X	(by construction)
and	PT is parallel to RL	(by construction)
so	PT is perpendicular to plane X	(Thm.4)

Q.E.F.

THEOREM 8: *Any plane containing a straight line perpendicular to another plane is itself perpendicular to that plane.*



Given: AB is perpendicular to plane X,
EHKG is a containing through AB

Prove: Plane EHKG is perpendicular to plane X.

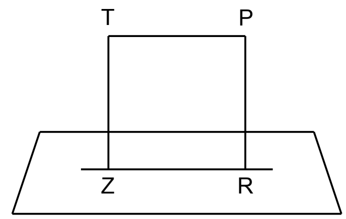
Choose any random point R on GK, the intersection of plane EHKG and plane X.
Draw RC perpendicular to GK in plane EHKG.
We already know that AB is also perpendicular to GK, since AB is perpendicular to all straight lines through B in plane X.

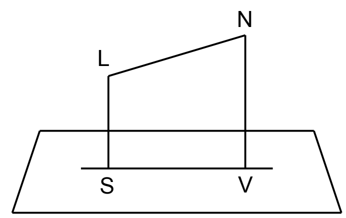
Since	AB is perpendicular to plane X	(given)
and	RC is parallel to AB	(RC and AB, in one plane, are ⊥ to GK)
thus	RC is perpendicular to plane X	(Thm.4)

For the same reasons, *any* straight line (in plane EHKG) drawn perpendicular to GK will be perpendicular to plane X. Therefore plane EHKG is perpendicular to plane X (Def. 3).

Q.E.D.

THEOREM 8 Remarks:



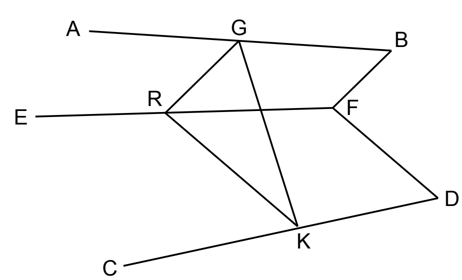


From this it is clear how to Drop a plane perpendicular to a given plane from a given straight line above the given plane, and how to Set up a plane perpendicular to a given plane upon a given straight line in the given plane.

Given a plane and a straight line in it, to set up a plane on that line perpendicular to the given plane: (1) pick any 2 points R and Z on the given line, (2) set up ZT and RP perpendicular to the given plane (Thm.7), (3) since ZT and RP are perpendicular to the same plane, therefore they are parallel (Thm.5), and thus are in one and only one plane together (Thm.2), (4) since the plane containing them passes through lines that are at right angles to the given plane, therefore their plane is at right angles to the given plane (Thm.8).

Given a plane and a straight line above it, to construct the plane which contains that line and is perpendicular to the base plane: (1) pick any 2 points L and N on the given line, (2) drop LS and NV perpendicular to the given plane (Thm.6). The rest of the proof is the same as above.

THEOREM 9: *If three straight lines are not all in one plane, and yet one of them is parallel to the other two, then the other two are also parallel to each other.*



Given: AB, CD, EF are three lines not all in one plane.

AB is parallel to EF.

CD is parallel to EF.

Prove: AB is parallel to CD.

[1] Choose R at random on EF.
Draw RG perpendicular to EF in the plane of parallels AB and EF.
Draw RK perpendicular to EF in the plane of parallels CD and EF.

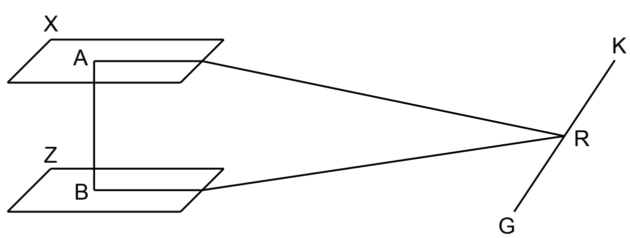
[2] Now R, G, K are all in one plane (Thm.1).
And since ER is perpendicular to both KR and RG in the plane of R, G, K, therefore ER is perpendicular to the plane of KRG (Thm.3).

[3]	Now	AG is parallel to ER	(given)
	and	<u>ER is perpendicular to plane K, R, G</u>	(Step 2)
	so	AG is perpendicular to plane K, R, G	(Thm.4)
[4]	But	CK is parallel to ER	(given)
	and	<u>ER is perpendicular to plane K, R, G</u>	(Step 2)
	so	CK is perpendicular to plane K, R, G	(Thm.4)

[5] Since AG and CK are both perpendicular to the same plane, namely the plane of points K, R, G, therefore AG and CK are parallel to each other (Thm.5).

Q.E.D.

THEOREM 10: *If one straight line is perpendicular to two planes, the planes are parallel.*



Given: AB is perpendicular to plane X and to plane Z.

Prove: Plane X is parallel to plane Z.

If possible, suppose planes X and Z are *not* parallel, but eventually meet each other – let KG be the line of their intersection. Pick point R at random on KG.

Join AR.
Join BR.

[1] Now, A and R are both in plane X, and so line AR is in plane X.
And B and R are both in plane Z, and so line BR is in plane Z.

[2] Since BA is perpendicular to plane X (given), therefore any straight line in plane X passing through A is at right angles to BA. But AR is in plane X (Step 1), and it passes through point A. Therefore AR is at right angles to BA.

Thus ∠BAR is right.

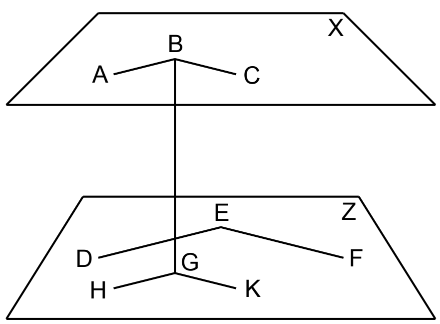
[3] Since AB is perpendicular to plane Z (given), therefore any straight line in plane Z passing through B is at right angles to AB. But BR is in plane Z (Step 1), and it passes through point B. Therefore BR is at right angles to AB.

Thus ∠ABR is right.

[4] Thus ABR is a triangle two of whose angles are right angles – which is impossible. Therefore our initial assumption was impossible, namely that planes X and Z should meet. Therefore planes X and Z never meet – and so they are parallel.

Q.E.D.

THEOREM 11: *If two intersecting lines in one plane are parallel to two intersecting lines in another plane, the two planes are parallel.*



Given: AB and BC intersect in plane X,
DE and EF intersect in plane Z,
AB is parallel to DE,
BC is parallel to EF.

Prove: Plane X is parallel to Plane Z

[1] Drop BG perpendicular to plane Z (Thm.6).
In plane Z, draw GH parallel to ED, and GK parallel to EF.

[2] Since BG is perpendicular to plane Z,
thus ∠BGH is right
and ∠BGK is right

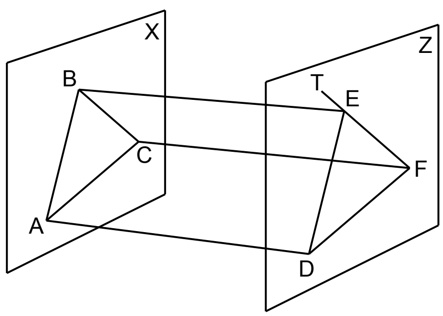
[3]	But	GH is parallel to DE	(we made it so)
	and	AB is parallel to DE	(given)
	so	AB is parallel to GH	(Thm.9)
	Thus	∠ABG is right	(since ∠BGH is right; Step 1)
[4]	Again	GK is parallel to EF	(we made it so)
	and	BC is parallel to EF	(given)
	so	BC is parallel to GK	(Thm.9)

Thus $\angle CBG$ is right (since $\angle BGK$ is right; Step 1)

[5] Therefore BG is at right angles to both AB and BC (Steps 3 and 4), which are two lines intersecting in plane X. Therefore BG is at right angles to plane X (Thm.3). But BG is at right angles to plane Z (we dropped BG at right angles to plane Z; Step 1). Therefore planes X and Z have a common perpendicular, namely BG, and thus these two planes are parallel to each other (Thm.10).

Q.E.D.

THEOREM 12: *A pair of intersecting lines parallel to another pair of intersecting lines in another plane will contain the same angle (or supplementary angles).*



Given: AB and BC intersect in plane X,
DE and EF intersect in plane Z,
AB is parallel to DE,
BC is parallel to EF.

Prove: $\angle ABC = \angle DEF$.

Cut off AB = DE, and cut off BC = EF.
Join AC, DF, AD, BE, CF.

[1] AB and DE are parallel (given), and so they are in one plane.
But we have just cut off AB and DE equal to each other.
Therefore the lines joining their endpoints are also parallel and equal (Ch.1).
i.e. AD and BE are parallel and equal to each other.

[2] BC and EF are parallel (given), and so they are in one plane.
But we have just cut off BC and EF equal to each other.
Therefore the lines joining their endpoints are also parallel and equal (Ch.1).
i.e. BE and CF are parallel and equal to each other.

[3]	Since	AD is parallel and equal to BE	(Step 1)
	and	<u>CF is parallel and equal to BE</u>	(Step 2)
	thus	AD is parallel and equal to CF	(Thm.9)

And so the lines joining their endpoints are also parallel and equal (Ch.1), i.e. AC is parallel and equal to DF.

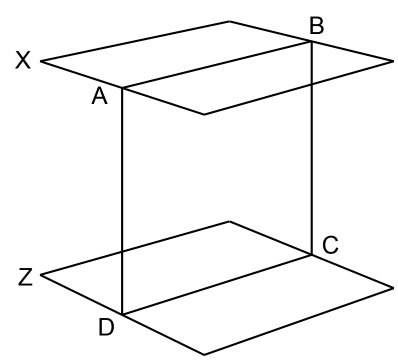
[4]	Now	AB = DE	(we cut them off equal)
	and	BC = EF	(we cut them off equal)
	and	AC = DF	(Step 3)
	thus	$\Delta ABC \cong \Delta DEF$	(Side-Side-Side)
	so	$\angle ABC = \angle DEF$	

Q.E.D.

THEOREM 12 Remarks:

If we extend FE to T, then TE is parallel to BC, too, but ∠TED will not be equal to ∠ABC (unless ∠TED and ∠FED are both right angles). Still, ∠TED is supplementary to ∠DEF, and therefore also supplementary to ∠ABC.

THEOREM 13: *If one plane intersects two parallel planes, the two lines of intersection are parallel.*



Given: Plane X is parallel to plane Z, each is cut by plane ABCD, namely at AB and CD.

Prove: AB is parallel to CD.

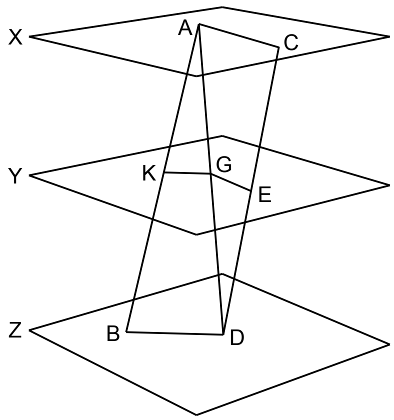
Since planes X and Z never meet in any direction, a line contained in one can never meet a line contained in the other. Therefore AB can never meet CD.

But since AB and CD are both in the one plane ABCD (given), therefore they are non-meeting straight lines *in the same plane*, and therefore they are parallel to each other.

So AB is parallel to CD.

Q.E.D.

THEOREM 14: *Straight lines cut by parallel planes are cut in the same ratios.*



Given: AB and CD are cut by three parallel planes X, Y, Z, cutting them off at A, K, B and C, E, D.

Prove: AK : KB = CE : ED.

[1] Join AD, AC, DB, GE, GK.

[2] Since A and C are both in plane X, thus AC is in plane X.
Since A and C are both in the plane of A, C, D, thus AC is in the plane of A, C, D.
Therefore AC is the line of intersection of plane X and plane A, C, D.

[3] Likewise EG is the intersection of plane Y and plane A, C, D.
GK is the intersection of plane Y and plane A, B, D.
DB is the intersection of plane Z and plane A, B, D.

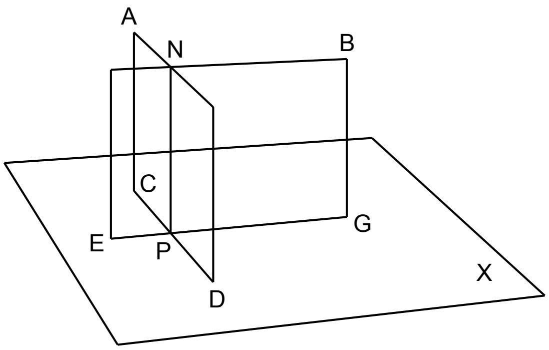
[4] Thus AC is parallel to EG, being intersections of plane ACD with the parallel planes X and Y. (Thm.13)

and GK is parallel to DB, being intersections of plane ABD with the parallel planes Y and Z. (Thm.13)

[5] And so, since GK is parallel to DB (Step 4) in ΔABD, thus
AK : KB = AG : GD
and since AC is parallel to EG (Step 4) in ΔACD, thus
CE : ED = AG : GD
and since in these two proportions two ratios are the same as a third, it follows that they are the same as each other, i.e.
AK : KB = CE : ED

Q.E.D.

THEOREM 15: *The intersection of two planes each perpendicular to a third plane is a straight line perpendicular to the third plane.*



Given: Planes A and B, both perpendicular to plane X, and intersecting each other along PN, P being in plane X.

Prove: PN is perpendicular to plane X.

[1] Since plane A is perpendicular to plane X, and CD is their intersection, therefore every line drawn in plane A perpendicular to CD is also perpendicular to plane X (Def. 3). Therefore the straight line drawn from P (in plane A), perpendicular to CD, is perpendicular to plane X.

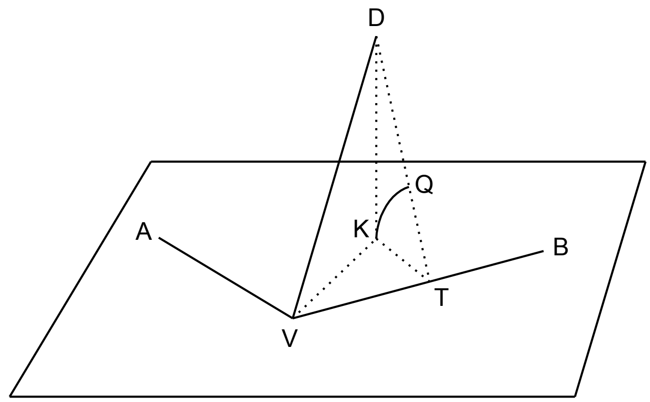
[2] Likewise since plane B is perpendicular to plane X, and EG is their intersection, therefore every line drawn in plane B perpendicular to EG is also perpendicular to plane X (Def. 3). Therefore the straight line drawn from P (in plane B), perpendicular to EG, is perpendicular to plane X.

[3] Therefore there is a perpendicular to plane X standing on point P *that lies in plane* A (Step 1), and again there is a perpendicular to plane X standing on point P *that lies in plane* B (Step 2). But there is only one perpendicular to plane X standing on point P (Thm.5 Remark). Therefore the line perpendicular to plane X, standing on point P, must be a line common to planes A and B. But the only line common to them is their line of intersection (Princ. 2), namely NP. Therefore the line perpendicular to plane X, standing on point P, is NP.

So PN is perpendicular to plane X.

Q.E.D.

THEOREM 16: *In a solid angle formed by three rectilineal angles, any two of those angles together are greater than the third.*



Let V be the vertex of a solid angle made up of the three rectilineal angles AVD, DVB, and AVB. I say that any two of these together are greater than the third.

[1] Drop DK perpendicular to the plane of AVB (Thm.6). In plane AVB, draw KT perpendicular to VB. Join DT.

[2]	Now since	DK is perpendicular to AVB,
	thus	plane DKT is perpendicular to plane AVB (Thm.8).
	So	any line in plane AVB that is perpendicular to KT (which is the intersection of planes DKT and AVB) must be perpendicular to plane DKT (Def.3).
	But	VT is perpendicular to KT (Step 1).
	Hence	VT is perpendicular to plane DKT.
[3]	Since	VT is perpendicular to plane DKT (Step 2),
	thus	VT is perpendicular to every line through T in plane DKT (Def.2).
	So	VT is perpendicular to DT, i.e. $\angle DTV$ is right.

[4] Now since DK is perpendicular to plane AVB, hence $\angle DKT$ is right.

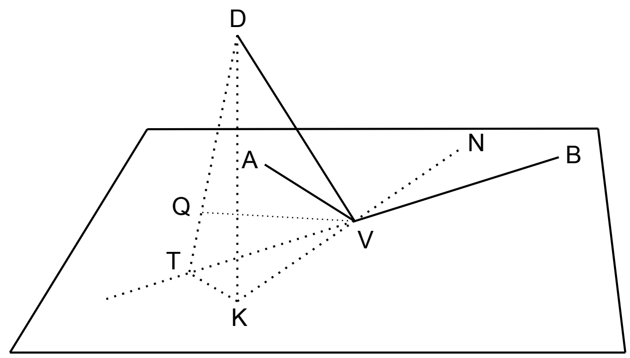
	Thus	$DT > TK$ (since DT is hypotenuse in right $\triangle DTK$)
	So cut off	$TQ = TK$.
	Now	$\angle VTK = \angle VTQ$ (both are right; $\angle VTQ$ is $\angle DTV$)
	and	<u>VT is common</u> (to triangles VTK and VTQ)
	so	$\triangle VTK \cong \triangle VTQ$
	so	$\angle QVT = \angle KVT$
[5]	Now	$\angle DVT > \angle QVT$ (the whole is greater than the part)
	so	$\angle DVT > \angle KVT$ ($\angle KVT = \angle QVT$, Step 4)
[6]	So	$\angle DVB > \angle KVB$ (Step 5)
	Similarly	$\underline{\angle DVA > \angle KVA}$
	hence	$\angle DVB + \angle DVA > \angle KVB + \angle KVA$ (adding)
	or	$\angle DVB + \angle DVA > \angle AVB$

So these two given angles are greater than the third. Since there was nothing special about the two angles we chose among the given three, it follows the same way that any two of them will be greater than the third.

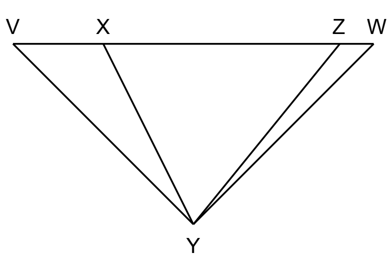
Q.E.D.

THEOREM 16 Remarks:

1. A solid angle contained by 3 plane angles is called a *trihedral angle*.

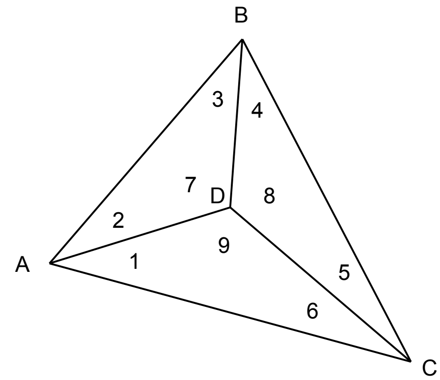


2. What if K lands outside angle AVB? Then the proof is identical up to Step 4, where we said ∠DVT > ∠KVT. Now extend KV through ∠AVB.
Thus ∠KVT = ∠NVB (vertical)
so ∠DVT > ∠NVB.
And since ∠DTV is right, hence ∠DVT is acute (in △DTV), and so its supplementary angle, ∠DVB, is obtuse.
Hence ∠DVB > ∠DVT
so ∠DVB > ∠NVB (since ∠DVT > ∠NVB above)
and ∠DVA > ∠NVA by the same reasoning. And the remainder of the proof is the same as in the Theorem.



3. To illustrate why this Theorem is true, draw any angle XYZ on a piece of paper, and on each side of it draw angles VYX and ZYW which together add up to an angle *less than* angle XYZ. Cut out △VYW, and fold along XY and YZ. Do triangles VYX and ZYW form a solid angle with triangle XYZ? Do they meet above the plane of △XYZ? What happens if ∠VYX + ∠ZYW = ∠XYZ?

THEOREM 17: *Any solid angle is contained by plane angles adding up to less than four right angles*.



Let's start once more with a "trihedral" angle, an angle formed by three plane angles, namely 7, 8, 9, all coming up to a point D. (You must imagine that point D is above the plane of this page.) I say that 7 + 8 + 9 is less than four right angles.

Choose A, B, C at random along the legs of the solid angle, and join AB, BC, CA, thus forming solid angles again at A and at B and at C. Looking at the diagram, then, you must remember that you are looking down like a bird at the peak of a solid pyramid – so ABC is the base of the pyramid, but angles 1 through 9 all lie in planes that rise up toward you from that base.

[1] Because A is a trihedral angle, thus $1 + 2 > \angle CAB$ (Thm.16)

[2] Because B is a trihedral angle, thus $3 + 4 > \angle ABC$ (Thm.16)

[3] Because C is a trihedral angle, thus $5 + 6 > \angle BCA$ (Thm.16)

[4] Adding together all these inequalities, keeping the greater things on one side,

$1 + 2 + 3 + 4 + 5 + 6 > \angle CAB + \angle ABC + \angle BCA$

but $\underline{\angle CAB + \angle ABC + \angle BCA = \text{two rights}}$ (triangle ABC)

so $1 + 2 + 3 + 4 + 5 + 6 > $ two rights

[5] Now angles 1 through 9, added together, equal all the angles in three triangles, and so all together they add up to three times the angle-sum of a triangle, i.e. three times two rights, i.e. six rights. So

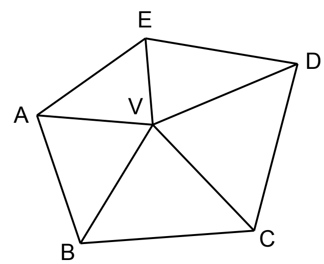
$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = $ six rights

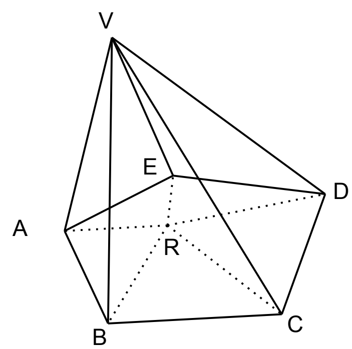
[6] Thus, if we subtract *more* than two rights from these nine angles, *less* than four rights will remain. But 1 + 2 + 3 + 4 + 5 + 6 is *more* than two rights (Step 4). Therefore, when subtracted from the nine angles, *less* than four rights remain, i.e.

$7 + 8 + 9 < $ four rights.

So the three plane angles forming a trihedral angle must add up to less than four right angles.

Q.E.D.





This Theorem is not limited to solid angles made of three plane angles. Take any solid angle with vertex V formed out of n plane angles. Pass a plane through the legs of the angle, forming a polygon base and a pyramid with vertex V. The polygon base will thus have n sides, and if we pick a random point R inside it, we can divide it into n triangles.

Now the angle-sum of the polygon base equals the angles of all those n triangles minus the angles around R, i.e. minus 360°. So the angles of the polygon $= (n \times 180° - 360°)$.

Since every vertex of the polygon base is also the vertex of a trihedral angle in the pyramid, hence very angle of the polygon must be less than the two angles above it which form the angles at the foot of the pyramid. For example, $\angle ABC < \angle ABV + \angle CBV$ (Thm.16). So all $2n$ angles about the foot of the pyramid add up to more than the n angles of the polygon, i.e. more than $(n \times 180° - 360°)$. So let those angles at the foot of the pyramid add up to $(n \times 180° - 360° + Z°)$.

Now the n plane angles forming the solid angle at V equal the angles in the n triangular faces of the pyramid *minus* their $2n$ angles at the foot of the pyramid. So the n angles forming solid angle V add up to

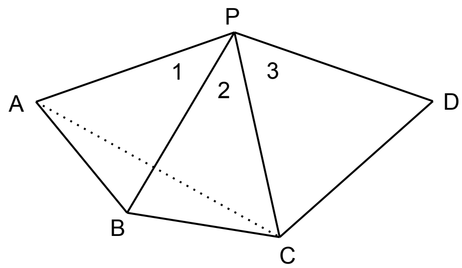
$$(n \times 180°) - (n \times 180° - 360° + Z°)$$

or

$$360° - Z°$$

So the n plane angles forming solid angle V add up to less than four rights.

THEOREM 18: *If among three angles in a plane any two are greater than the third, and they are made the peak angles of three isosceles triangles of the same leg-length, then likewise for the bases of these triangles, any two together will be greater than the third.*



Given: Three isosceles triangles whose legs are all equal, i.e. PA = PB = PC = PD, and whose peak angles (1, 2, 3) are such that any two are greater than the third.

Prove: Any two bases of these triangles will be greater than the third.

For example, I say that AB + BC > CD.

[1]	Join AC.		
[2]	Since	AP = CP = DP	(given)
	but	$\underline{\angle APC > \angle CPD}$	(given)
	thus	AC > CD	(Ch.1, Thm. 16 Question 1)
[3]	Now	AB + BC > AC	(triangle ABC)
	and	$\underline{AC > CD}$	(Step 2)
	so	AB + BC > CD	

Since there was nothing special about AB and BC, the same proof works just as well to show that BC + CD > AB, and again that AB + CD > BC. To show that AB + CD > BC, just rearrange the triangles so that angles 1 and 3 are next to each other, and 2 is on the outside.

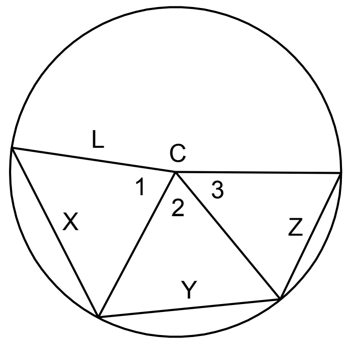
So whenever three isosceles triangles of the same leg-length are formed with three peak angles any two of which are greater than the third, likewise for their bases any two of them together will be greater than the third. *Q.E.D.*

THEOREM Remarks:

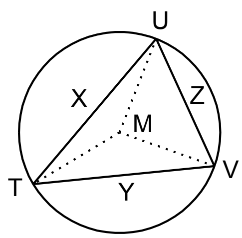
A quick corollary follows from this Theorem: we can make a triangle out of lengths AB, BC, CD, since any two of them are greater than the third. Thus we conclude: *When three isosceles triangles of the same leg-length are formed with three peak angles any two of which are greater than the third, then it will be possible to make a triangle out of the lengths of their bases.* For short, call such a triangle a "base triangle."

Obviously, this Theorem is simply a matter of plane geometry, but we will need it for the upcoming Theorem 20, here in solid geometry, where we shall construct a solid angle.

THEOREM 19: *If the peak angles of three isosceles triangles with a common leg-length* L *add up to less than four right angles, then* L *is greater than the radius of the circle circumscribing their "base triangle."*



Again, this is a matter of plane geometry, but it is crucial for the solid geometry in the next theorem. Start with three isosceles triangles of leg-length L, with peak angles 1, 2, 3 adding up to less than 360°, and bases X, Y, Z. Since they have the same leg length, L, if we place their equal sides together and give them a common vertex, C, the circle of center C and radius L will pass through the endpoints of bases X, Y, Z. Since 1 + 2 + 3 is less than 360°, hence the chords X, Y, Z do not cut off the circle's entire circumference.

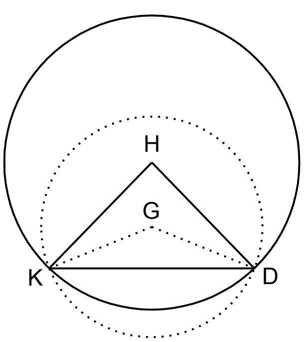


Now if the angles 1, 2, 3 are such that any two are together greater than the third, we can make a triangle out of X, Y, Z (Thm.18). So suppose this condition is met, and make △TUV with sides equal to X, Y, Z. Circumscribe a circle about △TUV (Ch.4). Call its center M.

Obviously the chords X, Y, Z together cut off the entire circumference of circle M. But these same chords together cut off only a portion of the circumference of circle C. Therefore circle C is greater than circle M, and so L (the radius of circle C) is greater than the radius of circle M.

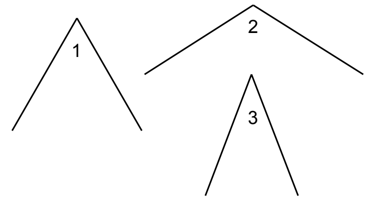
Q.E.D.

THEOREM 19 Remarks:



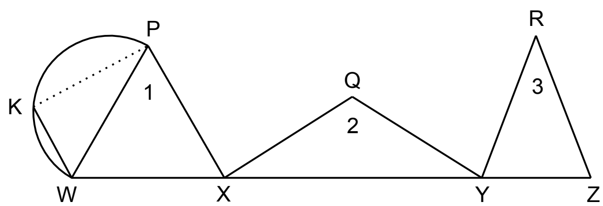
The proof takes it as evident that if the same chord length cuts off a greater portion of the circumference in one circle than it does in another, the other circle is greater than the one. For example, if KD cuts off an arc in circle G corresponding to ∠KGD, and an arc in circle H corresponding to ∠KHD, and ∠KGD > ∠KHD, then circle H is larger than circle G. To see it, compare isosceles triangles KGD and KHD. Since ∠KGD is greater than ∠KHD, the base angles of isosceles △KGD must be less than those of △KHD, and so KG and DG must meet *inside* △KHD. Hence the legs of △KGD are less than those of △KHD. So GK < HK, which means circle G is smaller than circle H.

THEOREM 20: *How to make a solid angle out of three plane angles. Thus it is required that they add up to less than four right angles, and that any two of them are greater than the third.*

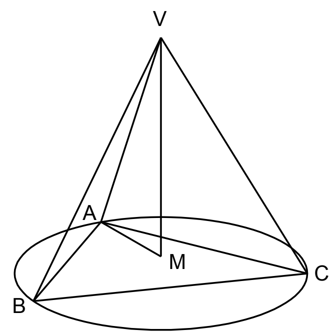


Let our three given plane angles be 1, 2, 3. By Theorem's 16 and 17 we know that it is impossible to make a solid angle out of them unless they meet the conditions that any two of them are greater than the third, and they add up to less than four right angles. So let them meet these conditions.

To make a solid angle out of them,



[1] Cut off any length PW along the leg of angle 1, and make three isosceles triangles PWX, QXY, RYZ, all having leg-length PW.



[2] Thus a triangle can be made out of their bases (Thm.18). So make triangle ABC with

AB = WX
and BC = XY
and CA = YZ.

Draw a circle around triangle ABC, find center M, and join MA.

[3] Draw a semicircle on PW. Setting your compass to length MA, make a circle (not shown) around center W, and where it cuts the semicircle call K. Thus WK = MA. This can be done because MA is less than diameter WP (by Thm.19).

[4] Join PK. Thus ∠PKW is right (Ch.3).
Set up MV perpendicular to the plane of the circle (Thm.7), making MV = PK.

[5]	Now	MV = KP	(we made it so; Step 4)
	and	MA = KW	(we made it so; Step 3)
	and	∠VMA = ∠PKW	(both are right; Step 4)
	so	△VMA ≅ △PKW	(Side-Angle-Side)
	thus	VA = PW	

Likewise VC and VB are also each equal to PW, the common leg-length of our original isosceles triangles.

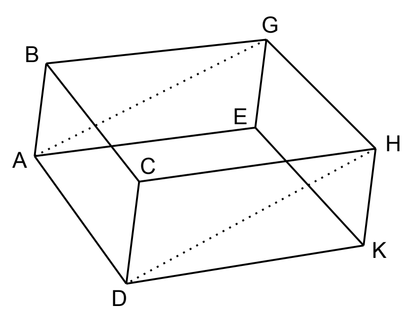
[6] But AB, BC, CA are equal to the bases of our isosceles triangles WX, XY, YZ (Step 2). So the three triangles standing on AB, BC, CA from point V are congruent to the three isosceles triangles (SSS), and hence the three peak angles forming solid angle V are equal to the given angles 1, 2, 3.

Q.E.F.

THEOREM 20 Remarks:

This Theorem is the converse of Theorems 16 and 17. In 16 and 17 we learned that any trihedral angle must be made of plane angles which add up to less than four rights and any two of which add up to more than the third one. But we were left wondering: are there *more* conditions required for three plane angles to be able to form a solid angle, or are those two conditions sufficient? Also, we might wonder this: the three angles must be less than four right angles – but do they in fact have to be less than *three* right angles, too? Or is it enough for them to be less than *four* right angles? This Theorem answers all those questions: as soon as the three plane angles are such that they are less than four right angles (by whatever amount you like), and such that any two of them are greater than the third, we can make them into a solid angle. Those conditions are not only necessary, but sufficient.

THEOREM 21: *If a solid is contained by three pairs of parallel planes, the opposite faces are congruent parallelograms (i.e. the solid is a parallelepiped).*



Suppose solid AH is contained by three pairs of parallel planes, namely BE and CK, and BH and AK, and BD and GK. I say that each pair of opposite faces, such as ABCD and EGHK, are identical parallelograms.

[1] Since AB and CD are the intersections of plane AC with the parallel planes BE and CK, therefore AB is parallel to CD (Thm.13).

[2] Since BC and AD are the intersections of plane AC with the parallel planes BH and AK, therefore BC is parallel to AD (Thm.13).

[3] Since AB is parallel to CD (Step 1)
and BC is parallel to AD (Step 2)
thus ABCD is a parallelogram.

Likewise the remaining 5 faces are parallelograms.

[4] Join AG, DH.

[5] Since AD is parallel to EK (because ADKE is a parallelogram)
and GH is parallel to EK (because GHKE is a parallelogram)
thus AD is parallel to GH (Thm.9)

[6] And thus A, D, G, H are all in one plane (Thm.2). And their plane intersects the parallel planes BE and CK at AG and DH, and therefore AG is parallel to DH (Thm.13). But AD was just proved parallel to GH (Step 5), and therefore AGHD is a parallelogram.

[7] So AG = DH (opp. sides in parallelogram AGHD)
and AB = DC (opp. sides in parallelogram ABCD)
and BG = CH (opp. sides in parallelogram BGHC)
so $\triangle ABG \cong \triangle DCH$ (Side-Side-Side)

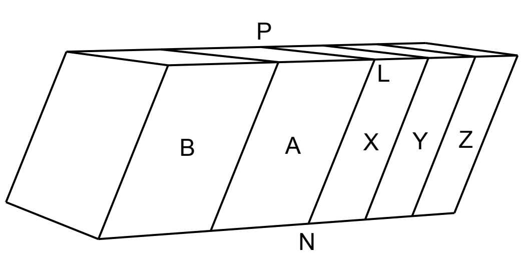
[8] But ABGE is just two of $\triangle ABG$, and DCHK is just two of $\triangle DCH$, similarly arranged. Therefore

ABGE ≅ DCHK.

Likewise the other opposite parallelograms containing the solid are congruent to each other.

Therefore if a solid is contained by 3 pairs of parallel planes, then its six faces are three pairs of congruent parallelograms, and such a solid is called a *parallelepiped*.

THEOREM 22: *If a parallelepiped is cut by a plane parallel to one of its pairs of opposite faces, the two resulting parts have to each other the same ratio as the bases on which they stand.*



Given: Parallelepiped A + X, cut by a plane at PLN parallel to one pair of its opposite faces, thus dividing it into two parallelepipeds, namely A and X.

Prove: volume of A : volume of X = area of base of A : area of base of X

[1] Place a solid B, identical to A, right next to it, and a solid Y, identical to X, right next to it. And thus multiply solids A and X however many times you like. Say you double A, and triple X.

[2] Because of the identical shape and size of solids A and B, it is clear that the *base* of the whole solid A + B is double the *base* of A.

Likewise the base of the whole solid X + Y + Z is triple the base of solid X.

[3] Now, because they lie inside the same parallels and have identical angles, if the *solid* A + B is equal in volume to the *solid* X + Y + Z, this can only be because they stand on equal bases, i.e. the *base* of A + B must be equal to the *base* of X + Y + Z.

But if the solid A + B is *bigger* than solid X + Y + Z, then A + B must stand on a bigger base than X + Y + Z does. And if the solid A + B is *smaller* than solid X + Y + Z, then A + B must stand on a smaller base than X + Y + Z does.

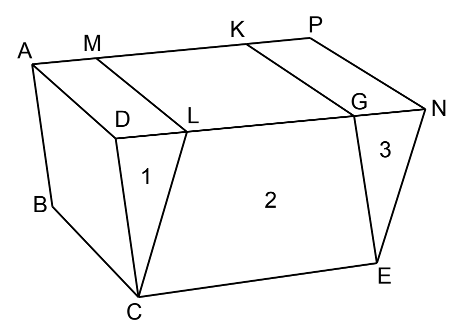
[4] Therefore, whatever multiple we take of solid A (and therefore of its base), and whatever multiple we take of solid X (and therefore of its base), the multiple solids must compare the same way as the corresponding multiple bases.

[5] Therefore solid A : solid B = base of A : base of B (Ch.5, Def.8)

Q.E.D.

THEOREM 23: *Parallelepipeds standing on the same base and having the same height are equal (i.e. they have the same volume).*

Given: Parallelepipeds AE and ME, both standing on base BCE and having their tops in the same plane.



Prove: AE and ME have the same volume.

First, suppose solids AE and ME not only have their tops in the same plane, but also that some other pair of their faces lie in the same plane, say CG and CN lie in the same plane – and therefore also the parallel faces BK and BP lie in the same plane on the opposite side.

But CDA and EGK do *not* coincide with CLM and ENP (if they did, the two solids would coincide entirely).

I say that solids AE and ME have the same volume.

[1]	For since CDGE and CLNE are both parallelograms, therefore		
		DG = LN	(each is equal to CE)
	so	DL = GN	(subtracting part LG from both sides)
	but	DC = GE	(in parallelogram CDGE)
	and	CL = EN	(in parallelogram CLNE)
	so	△DCL ≅ △GEN	(Side-Side-Side)

[2]	Now	AD is parallel to BC	(ABCD is a parallelogram)
	and	ML is parallel to BC	(BCLM is a parallelogram)
	so	AD is parallel to ML	(Thm.9)
	thus	ADLM is a parallelogram.	

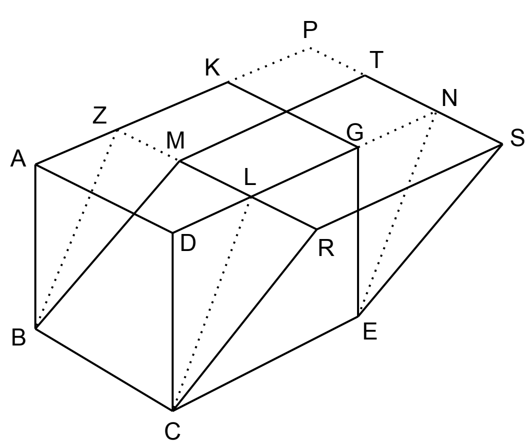
Clearly KGNP is also a parallelogram, and it is congruent to ADLM.
And, because they are opposite faces in the parallelepipeds,

AC and KE are congruent parallelograms

and MC and PE are congruent parallelograms

[3] Clearly, then, the two triangles and three parallelograms containing prism 1 are congruent with and arranged similarly to the two triangles and three parallelograms containing prism 3 (Steps 1 and 2). And thus they can be made to coincide and therefore have equal volumes.

[4]	So	prism 1 = prism 3	(Step 3)
	so	solid 1 + 2 = solid 2 + 3	(adding solid 2 to each side)
	i.e.	solid AE is equal to solid ME.	



Next, suppose that solids AE and ME have only their tops and bottoms in the same planes, and the front face of ME, namely CRSE, does *not* lie in the same plane as CDGE, the front face of solid AE.

AE and ME are still going to be equal in volume.

Let MRST be the top face of solid ME, in the same plane as ADGK, the top face of solid AE.

[1] Extend RM to Z on AK, and ST to P on the extension of AK.
Extend LG to N.
Join ZB, LC, NE, and P to X, the back corner of base BCE (which, to avoid cluttering up the diagram, I have not drawn).

[2] Now ZLNP is a part of the top plane, and the top plane is parallel to base BCEX (given). Thus
plane ZLNP is parallel to plane BCEX.

[3] And CLNE is a part of the face plane CDGE, which is parallel to the back plane BAKX (in solid AE). But BZPX is a part of the back plane. Thus
plane CLNE is parallel to plane BZPX.

[4] And ZLCB is a part of the side plane MRCB, which is parallel to the opposite side plane TSEX (in solid ME). But PNEX is a part of that opposite side plane. Thus
plane ZLCB is parallel to plane PNEX.

[5] Therefore the solid contained by planes
ZLNP and BCEX
and CLNE and BZPX
and ZLCB and PNEX
is contained by 3 pairs of parallel planes (Steps 2 – 3).
Therefore that solid, namely ZE, is a parallelepiped (Thm.21), and it stands on base BCEX and under the same height as the two given solids)

[6] Since solid ZE has its face CLNE in the same plane as CDGE, the face of solid AE, therefore solid ZE = solid AE, by the first part of this Theorem.

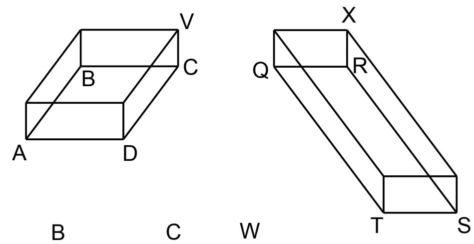
[7] Again, since solid ZE has its face ZLCB in the same plane as BMRC, the face of solid ME, therefore solid ZE = solid ME, by the first part of this Theorem.

[8] Therefore solid AE = solid ME (each being equal to solid ZE; Steps 6 and 7).

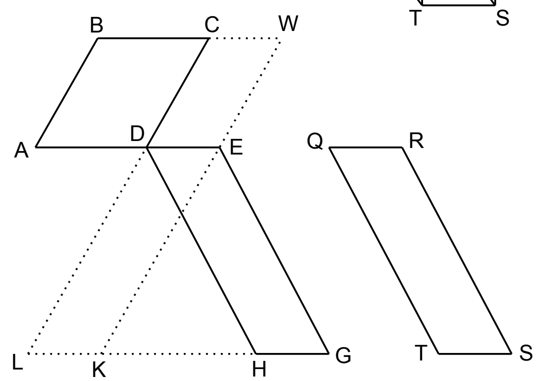
Therefore, no matter what, when two parallelepipeds have the same base and stand under the same height, they have the same volume.

Q.E.D.

THEOREM 24: *Parallelepipeds which are of the same height and on bases of equal area are equal.*



Conceive two parallelepipeds, AV and TX, with the same height and with bases ABCD and QRST having the same area. I say the solids have the same volume.



[1] Let's take the simplest case first: let the sides of these solids all be perpendicular to their bases – thus CV and RX are perpendicular to the bases and CV = RX (because the heights are the same). Because the walls of these solids are thus all standing at right angles to the bases, we can imagine the solids like two buildings, and just look at their "floor plans," namely their bases ABCD and QRST.

Now, to prove that AV = TX …

[2] We place a solid identical to TX in line with AD, that is, letting DEGH (identical to base QRST) be its base, we place DE in a straight line with AD. Complete parallelogram CDEW in the base plane, and build a "building" on it with the same height again as the solids on ABCD and DEGH.

[3] Extend CD to where it meets GH extended, namely at L, and complete parallelogram EDLK in the base plane, and build another "building" on top of it with the same height once more.

[4] Now, there is an undrawn rectangle standing straight up on DE (coming up at you out of the page) which is a *wall* for the building on DEKL; but it is also a wall for the building on DEGH. Since there is no absolute up and down in geometry, this wall can also be thought of as a base of each of these two solid buildings, and both are under the same height, i.e. both their tops lie in the plane standing on LKHG. Therefore they are equal in volume (Thm.23).

So The solid on DEKL = the solid on DEGH

[5] Notice that the buildings on DEKL and CDEW together make up one big parallelepipedal building, since they are in line with each other. Therefore, by Thm.22,

building on DEKL : building on CDEW = area of DEKL : area of CDEW,

[6] But, looking just at the parallelograms in the base plane,

	DEKL = DEGH	(both stand on DE, and are in the same parallels)
but	DEGH = QRST	(we made DEGH identical to QRST)
and	QRST = ABCD	(given)
so	DEKL = ABCD	

[7] So, substituting ABCD for DEKL in the proportion from Step 5, we have:

building on DEKL : building on CDEW = area of ABCD : area of CDEW,
But also by Theorem 22, we have

building on ABCD : building on CDEW = area of ABCD : area of CDEW

Since we have two ratios the same as a third ratio, they are the same as each other, i.e.

blding on DEKL : blding on CDEW = blding on ABCD : blding on CDEW.

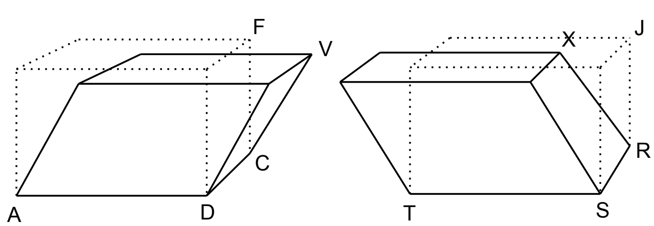
Notice in this proportion the buildings on DEKL and ABCD both have the same ratio to the building on CDEW. From this, it follows that they are equal. Thus

building on DEKL = building on CDEW.

[8]	Now	solid on DEKL = solid on ABCD	(Step 7)
	but	solid on DEKL = solid on DEGH	(Step 4)
	so	solid on ABCD = solid on DEGH	
	but	solid on QRST = solid on DEGH	(we made it thus in Step 2)
	so	solid on ABCD = solid on QRST	

Therefore the solid AV is equal in volume to the solid TX.

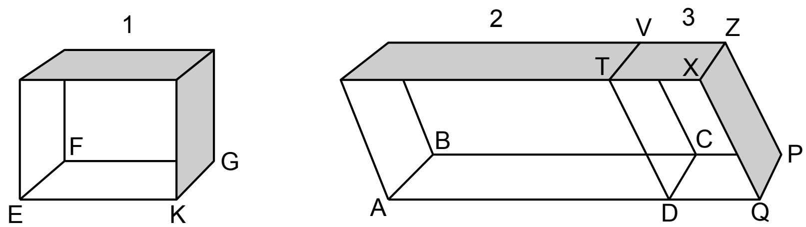
[9] Now what if the solids on ABCD and QRST, although having their tops and bottoms in the same planes, yet have their walls tilted in different ways? Will they still be equal? Yes.



Just build the solids on those same bases whose walls *are* perpendicular to the bases, having their tops also in the same top-plane as the "tilty" solids. Then, by Theorem 23, each upright solid is equal to the tilty solid whose base it shares. But, by the proof we just gave, the two upright solids are equal to each other – since they stand on equal bases and between the same parallel planes. Therefore the tilty solids are equal, too.

Q.E.D.

THEOREM 25: *Parallelepipeds of the same height are to each other as their bases.*



Given: Parallelepipeds 1 and 2 of the same height, standing on bases EFGK and ABCD.

Prove: Solid 1 has to solid 2 the same ratio that base EFGK has to base ABCD.

[1] Extend base ABCD so that parallelogram DCPQ, while having the same angles as parallelogram ABCD, nonetheless has the same *area* as EFGK.

[2] Complete the parallelepipedal solid on DCPQ by extending the planes of solid 2, and by capping it off with plane QXZP parallel to plane DTVC. Thus we have solid 3, and solids 2 and 3 together form one big parallelepiped.

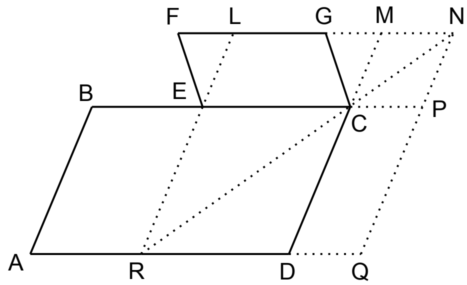
[3] Now solid 3 : solid 2 = DCPQ : ABCD (Thm.22)

[4] But solid 3 = solid 1,
since they stand between the same parallel planes, and have bases of equal area (Thm.24). Substituting solid 1 for solid 3 in the proportion from Step 3, then, we have:
solid 1 : solid 2 = DCPQ : ABCD

[5] But DCPQ = EFGK, by Step 1. Substituting EFGK for DCPQ in the proportion, we now have
solid 1 : solid 2 = EFGK : ABCD,
which is what we sought to prove.

Q.E.D.

THEOREM 25 Remarks:



You might be wondering how we accomplish Step 1. How do we extend the base ABCD with a parallelogram DCPQ that is equiangular with ABCD, but equal in area to EFGK?

Since that all takes place in the base plane, it is a matter of simple plane geometry, and Chapter 1 gives us all we need:

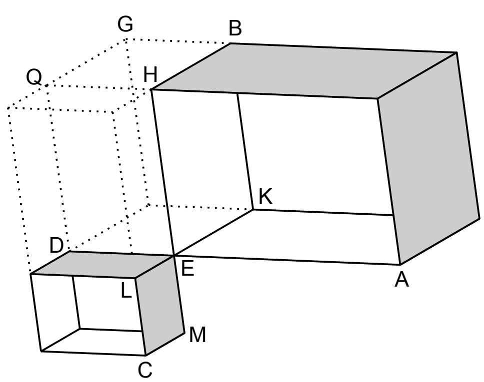
[1] Place EFGK on BC so that K is on point C.

[2] Draw LER parallel to AB and CD. Join RC. Extend RC and FG until they meet at N. Extend DC to M. Complete parallelogram DMNQ. Extend BC to P.

[3] Parallelogram DCPQ is clearly equiangular with parallelogram ABCD. But it is equal to parallelogram EFGC in area,

since	DCPQ = ECML	(complements in parallelogram RLNQ)
and	EFGC = ECML	(in the same parallels and on the same base)
so	DCPQ = EFGC	

THEOREM 26: *Similar parallelepipedal solids are to one another in the triplicate ratio of their corresponding sides.*



Given: Similar parallelepipeds AB and CD, with sides AE and ED being a pair of corresponding sides.

Prove: Solid AB : solid CD is the ratio triplicate of AE : ED.

[1] Place solids AB and CD so that they have a common corner at E, and the corresponding sides AE and ED lie in a straight line. Thus the corresponding sides LE and EK will also line up (since ∠LED = ∠KEA in the similar solids).

[2] In angles HED and HEK complete parallelepiped EG.
In angles HED and HEL complete parallelepiped LQ.

[3] Because of the similarity of the solids, AE, KE and HE are proportional to ED, EL, and EM. Hence

AE : ED = KE : EL = HE : EM

[4] Now, because parallelograms under the same height are to one another as their bases (Ch.6, Thm.1), it follows that:

AE : ED = AK : KD
and KE : EL = KD : DL
and HE : EM = HD : DM.

Because of Step 3, the first in each of these pairs of ratios are all the same ratio. Therefore the second in each of these pairs of ratios are also all the same,
i.e. AK : KD = KD : DL = HD : DM.

[5] But since parallelepipeds under the same height are to each other as their bases (Thm.25), it follows further that:

AK : KD = solid AB : solid EG
and KD : DL = solid EG : solid LQ
and HD : DM = solid LQ : solid CD

Because of Step 4, the first in each of these pairs of ratios are all the same ratio. Therefore the second in each of these pairs of ratios are also all the same,
i.e. solid AB : solid EG = solid EG : solid LQ = solid LQ : solid CD

[6] Since that proportion is continuous, and contains four terms, therefore the first has to the last the triplicate ratio of the first to the second, i.e.

solid AB : solid CD is the triplicate ratio of solid AB : solid EG.

[7] But, as we saw above in Steps 5 and 4,

solid AB : solid EG = AK : KD = AE : ED.
Therefore solid AB : solid CD is the triplicate ratio of AE : ED.

So similar parallelepipeds have to each other the triplicate ratio of their corresponding sides.

Q.E.D.

THEOREM 26 Remarks:

The most important instance of this, of course, is with cubes. All cubes are similar parallelepipeds, and so it follows that they are to each other in the ratio triplicate of their corresponding sides.

For example, suppose you had a pair of cubes, and the side or edge of one was *double* the side or edge of the other, i.e. their sides were in the ratio of 1 : 2. Then what is the ratio of their volumes? It will be 1 : 8, since

1 : 2 = 2 : 4 = 4 : 8,

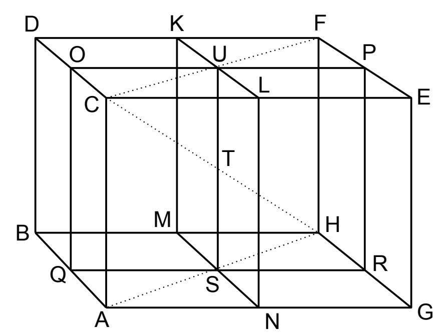
and thus 1 : 8 is the ratio triplicate of 1 : 2.

This Theorem should make you wonder about the ratios of other kinds of similar solids, such as curved ones. Do spheres have to each other the triplicate ratio of their diameters?

THEOREM 27: *If the sides of opposite faces in a parallelepiped are bisected by two planes, then the intersection of these two planes bisects (and is bisected by) the diagonal of the solid.*

Given: Parallelepiped BE, with diagonal CH. Planes QOPR and MKLN bisect the edges at Q, O, M, K, R, P, N, L. SU is the intersection of these two cutting planes.

Prove: SU and CH bisect each other.



[1] Join CU, UF.

[2] It is easily seen that OULC and UPEL are parallelograms.

Thus	OU = CL	
and	UP = LE.	
but	<u>CL = LE</u>	(given)
thus	OU = UP	
but	OC = PF	(being halves of the equal sides DC and EF)
and	<u>∠UOC = ∠UPF</u>	(each is equal to ∠PEL)
so	△UOC ≅ △UPF	(Side-Angle-Side)
thus	∠OUC = ∠PUF.	

[3]	But OUP is a straight line, and therefore CUF is also a straight line, since the vertical angles OUC and PUF are equal.

Likewise ASH is a straight line.

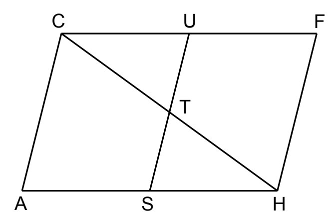
And since AC and FH are equal and parallel lines, ACFH is a parallelogram.

[4]	Thus SU lies in the plane of parallelogram ACFH, since it joins points U and S which lie on its opposite sides. Thus CH and SU must meet, say at T.

[5]	Now	CU = UF	(since △UOC ≅ △UPF; Step 2)
and	AS = SH	(since similarly △SQA ≅ △SRH)
Thus SU joins the midpoints of the opposite sides in parallelogram ACFH. Therefore SU bisects the diameter of ACFH, namely CH, and also is bisected by it.

Q.E.D.

THEOREM 27 Remarks:



1.	If it is not perfectly clear why the line joining the midpoints of a parallelogram's opposite sides must bisect and be bisected by the diagonal, consider the following. Let ACFH be a parallelogram, and let CU = UF, and AS = SH.

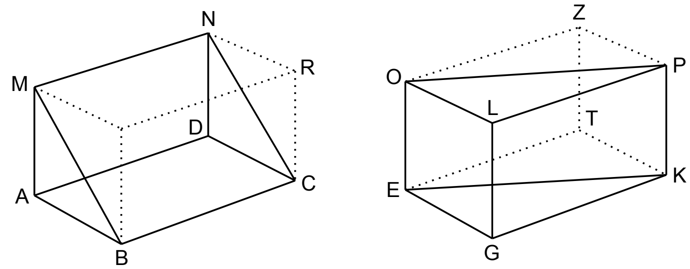
Now	CU = SH	being halves of the opposite sides of a parallelogram,
and	∠HCF = ∠CHS	since CF is parallel to HA
and	∠CUS = ∠HSU	since CU is parallel to SH
so	△CUT ≅ △HST	(Angle-Side-Angle)
so	UT = TS	
and	CT = TH	*Q.E.D.*

2.	Obviously, this Theorem is true about cubes in particular – if the sides of a cube are bisected by two planes, the intersection of those planes will bisect the diagonal of the cube, and be bisected by it.

3.	In parallelepipeds other than cubes, the four diagonals can be unequal to each other. But that doesn't make any difference to this Theorem – take any diagonal you like, the proof did not require that we choose a special one.

THEOREM 28: *If a triangular prism lies on one of its parallelogram sides, and in this position has the same height as another triangular prism lying on its triangular base, and if the parallelogram is double the triangle, then the prisms will have the same volume.*

Imagine a prism with triangular bases ABM and DCN, lying on one of its parallelogrammic sides ABCD, and another prism with triangular bases EGK and OLP, lying on EGK, which has half the area of ABCD.



Now if we further suppose that ABCD and EGK lie in the same plane, and also that OLP and MN lie in the same plane, then I say that the prisms will have the same volume.

[1] Complete the parallelepiped AR contained by the angles ADC, ADN, NDC.
Complete parallelogram EGKT, and
Complete the parallelepiped GZ contained by the angles GKT, GKP, PKT.

[2] Since ABCD is double triangle EGK in area, and EGKT is also double triangle EGK in area, therefore ABCD = EGKT.

[3] But that means that solids AR and GZ stand on equal bases. And yet they also have the same height, since it is given that the height of the prisms is the same, and we made the parallelepipeds to have that same height. Therefore AR and GZ have the same volume (Thm. 24).

[5] Since AR and GZ are the same volume, therefore also their halves have the same volume. But the triangular prisms are obviously their halves. Therefore the two prisms are equal in volume, too.

Q.E.D.

THEOREM 28 Remarks:

We assumed in this Theorem that each prism is obviously half the volume of the parallelepiped of which it is a part. Why is that obvious?

Consider the prism contained by triangles OLP and EGK. It makes up a parallelepiped by being combined with another prism, the one contained by triangles OZP and ETK. Now EGKT and OLPZ are parallelograms, and so are OZTE and all the other faces of the parallelepiped.

Thus $\triangle OLP \cong \triangle OZP$

and $\triangle EGK \cong \triangle ETK$

and LPKG ≅ OZTE

and OLGE ≅ ZPKT

and, of course, OPKE is a common face for both prisms.

So the two prisms are contained by an equal number of congruent and similarly arranged faces. Therefore they are congruent and contain equal volumes.

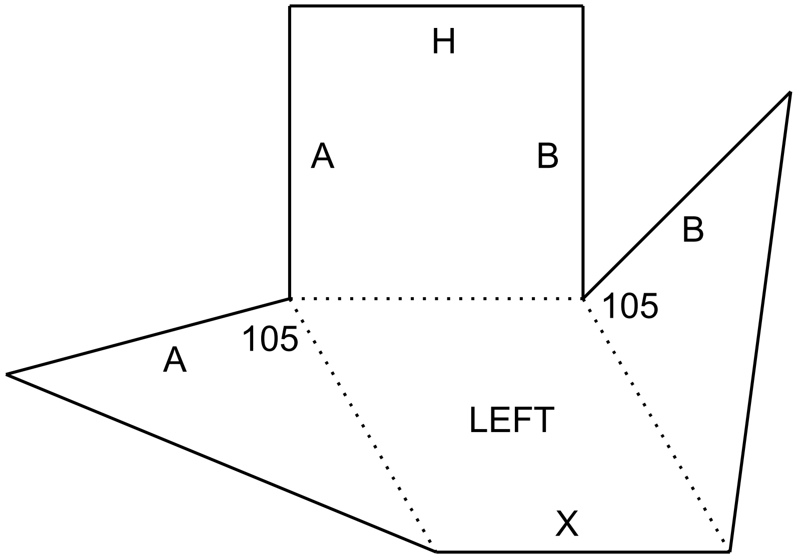
Does that mean that these prisms can *coincide*? Not necessarily.

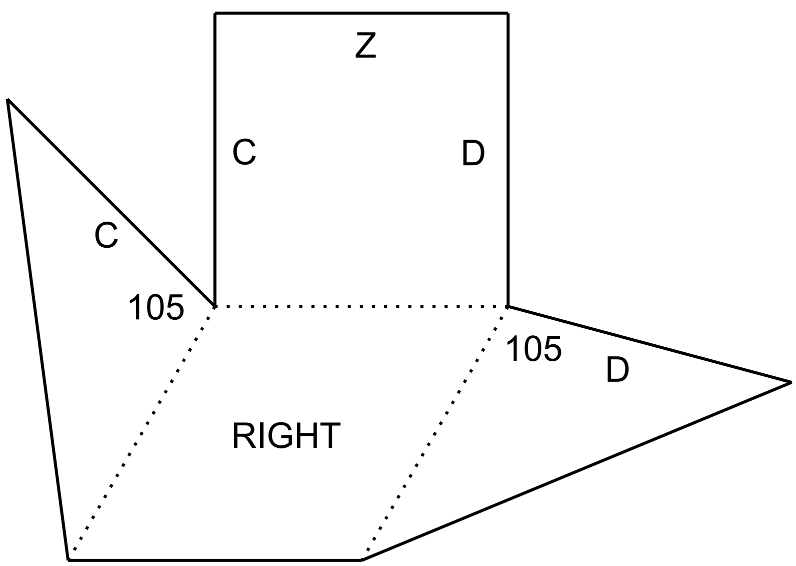
Consider your right hand and your left hand. Even if they were perfectly symmetrical, and of a ghostly quality so that they could pass through each other, they would not be able to coincide with each other and form one self-same hand. A right hand simply can't be a left hand!

Now, can the two prism halves of a parallelepiped be like that? Can they be perfect mirror images of each other, and yet not be able to coincide? Yes. It is almost impossible to represent this in a two-dimensional diagram in a clear and convincing way, so the best thing to do is to *make* a pair of such prisms. It is best not to use paper, since that is too flimsy – you need something more rigid like cardstock or a manila folder. Transfer the diagrams below onto a piece of manila: each consists of a square, a rhombus with angles of 60° and 120° (it is made of two equilateral triangles), and two isosceles triangles with peak angles of 105° (i.e. 60° + 45°) placed at the bottom corners of the square. The legs of the isosceles triangles are equal to the sides of the square.

After you have transferred the diagrams, cut out the two figures along the *solid* lines. Next, with all the labeling face up on the table, fold up the triangles and square along all the *dotted* lines. Bring together the edges marked with the same letters, such as "A", and tape them together. When you are done, you will have two triangular prisms, each with one open face. If you place the square faces down on the table and turn H and Z toward you, you will see that the prisms are symmetrical, but, like a right hand and a left hand, cannot be made to coincide. Their corresponding faces can be made to coincide one at a time, but not all of them simultaneously. If you pick them up in your hands, and place edges X and Z together, and in that position bring together the two open faces of the prisms, you will be holding a parallelepiped.

What makes the equality of these two prisms obvious, then, is not that they could be made to coincide. Rather, like your two hands, it is their perfect symmetry – one is a perfect mirror image of the other.


H
A
B
B
105
105
A
LEFT
X


Z
C
D
C
105
105
D
RIGHT

“HOOK”: TRIANGULAR SECTIONS OF A CUBE.

If you are given a cube and a triangle abc, will it be possible to slice the cube with a plane so that there will be formed a triangular facet which is similar to abc? Not if abc is right or obtuse. But if abc is acute, it can always be done.